

Bicyclic graphs with extremal degree resistance distance

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Abstract

Let $r(u, v)$ be the resistance distance between two vertices u, v of a simple graph G , which is the effective resistance between the vertices in the corresponding electrical network constructed from G by replacing each edge of G with a unit resistor. The degree resistance distance of a simple graph G is defined as $D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]r(u, v)$, where $d(u)$ is the degree of the vertex u . In this paper, the bicyclic graphs with extremal degree resistance distance are strong-minded. We first determine the n -vertex bicyclic graphs having precisely two cycles with minimum and maximum degree resistance distance. We then completely characterize the bicyclic graphs with extremal degree resistance distance.

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1 Introduction

Topological indices based on the various distances between the vertices of a graph are widely used in theoretical chemistry to establish relations between the structure and the properties of molecules. They provide correlations with physical, chemical and thermodynamic parameters of chemical compounds [5, 13, 35, 39, 40, 41, 42, 44].

The graphs considered in this paper are finite, loopless, and containing no multiple edges. Given a graph G , let $V(G)$ and $E(G)$ be, respectively, its vertex and edge sets. The ordinary distance $d(u, v) = d_G(u, v)$ between the vertices u and v of the graph G is the length of the shortest path between u and v [2]. The Wiener index [14] of a connected graph G , denoted by $W(G)$, is defined as the sum of all distances between unordered pairs of vertices u and v , i.e.,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v).$$

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The resistance distance between the vertices u and v of the graph G , denoted by $r(u, v)$, is defined to be the effective resistance between the nodes u and v in G . Analogous to Wiener index, the Kirchhoff index is defined as [7, 15]

$$Kf(G) = \sum_{\{u,v\} \subseteq V(G)} r(u, v)$$

which has been widely studied [3, 6, 8, 9, 10, 12, 16, 21, 22, 24, 25, 26, 27, 28, 29, 31, 32, 36, 38, 42, 43, 45]. For other notations in graph theory, we follow [2] and recent papers [17, 18, 19, 20, 23, 37].

A modified version of the Wiener index is the degree distance defined as

$$D(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]d(u, v),$$

where $d(u) = d_G(u)$ is the degree of the vertex u of the graph G . It was introduced independently by Gutman [11], Dobrynin and Kochetova [6] as a weighted version of the Wiener index.

Analogous to the definition of the degree distance, the degree resistance distance is defined as

$$D_R(G) = \sum_{\{u,v\} \subseteq V(G)} [d(u) + d(v)]r(u, v),$$

which was introduced by I. Gutman, L. Feng and G. Yu in [12]. Palacios [30] named the same graph invariant “additive degree-Kirchhoff index” and gave tight upper and lower bounds for the degree resistance distance of a connected undirected graph by using Markov chain theory.

Tomescu [34] determined the unicyclic and bicyclic graphs with minimum degree distance $D(G)$. In [33], the author investigated the properties of connected graphs having minimum degree distance $D(G)$. Bianchi et al. [1] gave some upper and lower bounds for degree resistance distance $D_R(G)$ whose expressions do not depend on the resistance distances. Recently, Chen et al. [4] characterized the maximal degree resistance distance of unicyclic graphs. In a natural way, we will consider the bicyclic graphs. A bicyclic graph is a connected graph whose edge number is one more than its vertex number. Obviously a bicyclic graph contains either two or three cycles. Throughout this article, we restrict our consideration on bicyclic graphs with exactly two cycles.

For convenience, let $\mathcal{B}_n^{p,q}$ be the set of the bicyclic graphs with exactly two cycles C_p and C_q as follows: $C_p = v_1v_2 \cdots v_pv_1$ and $C_q = u_1u_2 \cdots u_qu_1$ are two cycles such that there is a path $P = v_1w_1 \cdots w_{m-1}u_1$ joining them. The trees T_{v_i} , T_{u_j} and T_{w_k} are rooted at v_i , u_j and w_k , respectively. We say a tree T trivial if $|V(T)| = 1$, i.e., T is an singleton vertex, see Fig. 1.

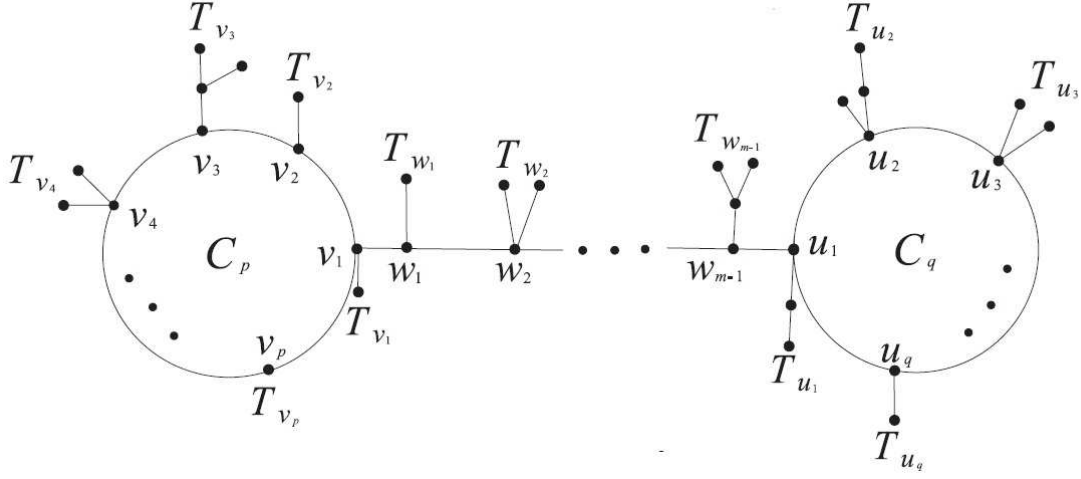


Fig. 1. Illustration for a graph in graph $\mathcal{B}_n^{p,q}$.

Let $S_n^{p,q}$ be the graph obtained from cycles C_p and C_q by attaching $n + 1 - p - q$ pendent edges to the unique common vertex of them. Let $P_n^{p,q}$ be the graph consisting of two disjoint cycles C_p and C_q and a path of length $n - p - q + 1$ joining them. If $n - p - q + 1 = 0$, $P_n^{p,q}$ coincides with $S_n^{p,q}$, see Fig. 2.

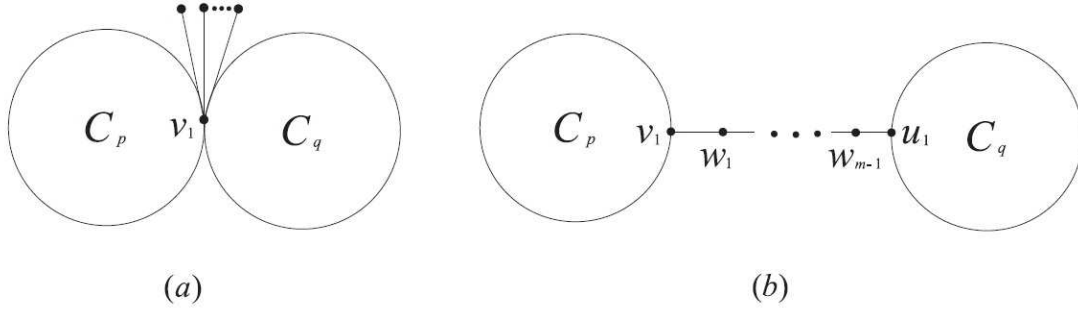


Fig. 2. (a) $S_n^{p,q}$ and (b) $P_n^{p,q}$.

To the best of our knowledge, the extremal degree resistance distances for bicyclic graphs has not been considered so far. In this paper, we firstly characterize n -vertex bicyclic graphs with exactly two cycles having minimum and maximum degree resistance distance, and then characterize the bicyclic graphs with extremal degree resistance distance.

2 Preliminaries

In this section, we provide some lemmas and graph transformations, that play an important role in the subsequent sections. Let $r(u, v)$ denote the resistance distance between u and v in the graph G . Recall that $r(u, v) = r(v, u)$ and $r(u, v) \geq 0$ with equality if and only if $u = v$.

In what follows, for the sake of simplicity, instead of $u \in V(G)$ we write $u \in G$. For a vertex v in G , we define

$$Kf_v(G) = \sum_{u \in G} r(u, v) \quad \text{and} \quad D_v(G) = \sum_{u \in G} d(u)r(u, v).$$

The definition of $D_v(G)$ implies that

$$D_R(G) = \sum_{v \in G} d(v) \sum_{u \in G} r(u, v).$$

For a vertex $v \in G$, $G - v$ denotes the graph obtained from G by deleting v and its incident edges. Let \overline{G} be the complement of G . Let $G - e$ ($G + e$) denote the graphs obtained from G by deleting (or adding respectively) the edge e .

Let H be a subgraph of graph G . For a vertex $u \in V(H)$, let

$$r(u|H) = \sum_{v \in V(H)} r(v, u|H), \quad S'(u|H) = \sum_{v \in V(H)} d(v)r(v, u|H).$$

If C_n is a cycle with the vertex set $V(C_n) = \{v_1, \dots, v_n\}$ and $n \geq 3$, by Ohm's law, we have that, for $v_i, v_j \in V(C_n)$ with $i < j$,

$$r_{C_n}(v_i, v_j) = \frac{(j-i)(n+i-j)}{n}.$$

Lemma 2.1 ([15]). *Let x be a cut-vertex of a connected graph, and let a and b be the vertices occurring in different components which arise upon deletion of x . Then*

$$r(a, b) = r(a, x) + r(x, b).$$

Lemma 2.2 ([12]). *Let C_k be the cycle of size k , and $v \in C_k$. Then*

$$Kf(C_k) = \frac{k^3 - k}{12}, \quad D_R(C_k) = \frac{k^3 - k}{3}, \quad Kf_v(C_k) = \frac{k^2 - 1}{6}, \quad D_v(C_k) = \frac{k^2 - 1}{3}.$$

Lemma 2.3 ([12]). *Let G_1 and G_2 be connected graphs with disjoint vertex sets, with n_1 and n_2 vertices, and with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1)$, $u_2 \in V(G_2)$. Constructing the graph G by identifying the vertices u_1 and u_2 , and denote the so obtained vertex by u . Then*

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2r(u_1|G_1) + 2m_1r(u_2|G_2) + (n_2 - 1)S'(u_1|G_1) + (n_1 - 1)S'(u_2|G_2).$$

Definition 2.1 *Let G be a bicyclic graph G with $V(G) = \{u, v, v_1, \dots, v_s\} \cup V(C_p) \cup V(C_q)$, for which v is a vertex of degree $s + 1$ such that vv_1, vv_2, \dots, vv_s are pendent edges incident with v , and u is the neighbor of v distinct from v_i that is on the cycle C_q . The other cycle C_p only has one common vertex w with C_q . We form a graph $G' = \sigma(G, v)$ by deleting the edges vv_1, vv_2, \dots, vv_s and adding new edges uv_1, uv_2, \dots, uv_s . We say that G' is a σ -transform of the graph G , see Fig. 3.*

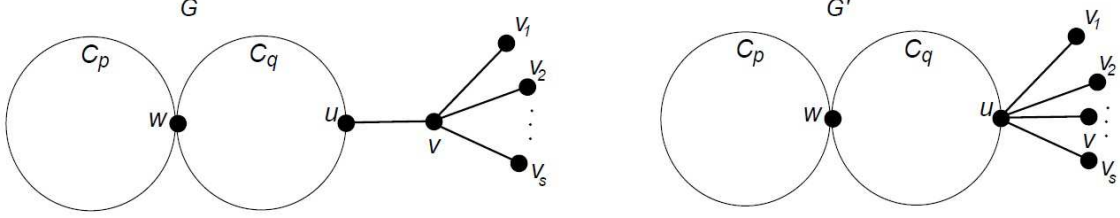


Fig. 3. σ -transform of a vertex v .

Lemma 2.4 Consider the graphs defined in Definition 2.1 and let $G' = \sigma(G, v)$ be a σ -transform of the bicyclic graph G , for convenience, let $r(w, u) = t$. Then $D_R(G) > D_R(G')$.

Proof. Let H be a subgraph $G[\{u, v, v_1, \dots, v_s\} \cup V(C_q)]$ and $T = G[\{u, v, v_1, \dots, v_s\}]$ (H' be a subgraph $G'[\{u, v, v_1, \dots, v_s\} \cup V(C_q)]$ and $T' = G'[\{u, v, v_1, \dots, v_s\}]$, respectively). Note that H and C_p (H' and C_p , respectively) share a common vertex w . By Lemmas 2.2 and 2.3, we get

$$D_R(G) = D_R(C_p) + D_R(H) + 2(s+1+q)r(w|C_p) + 2pr(w|H) + (q+s)S'(w|C_p) + (p-1)S'(w|H),$$

$$D_R(H) = D_R(C_q) + D_R(T) + 2(s+1)r(u|C_q) + 2qr(u|T) + (s+1)S'(u|C_q) + (q-1)S'(u|T),$$

$$D_R(G') = D_R(C_p) + D_R(H') + 2(s+1+q)r(w|C_p) + 2pr(w|H') + (q+s)S'(w|C_p) + (p-1)S'(w|H'),$$

$$D_R(H') = D_R(C_q) + D_R(T') + 2(s+1)r(u|C_q) + 2qr(u|T') + (s+1)S'(u|C_q) + (q-1)S'(u|T'),$$

$$D_R(G) - D_R(G') = 2qr(u|T) + (q-1)S'(u|T) + 2pr(w|H) + (p-1)S'(w|H)$$

$$- 2qr(u|T') - (q-1)S'(u|T') - 2pr(w|H') - (p-1)S'(w|H')$$

$$= 2q(2s+1) + (q-1)(s+1+2s) + 2p\left[\frac{q^2-1}{6} + t+1 + (t+2)s\right]$$

$$+ (p-1)\left[\frac{q^2-1}{3} + (t+1)(s+1) + (t+2)s\right] - 2q(s+1) - (q-1)(s+1)$$

$$- 2p\left[\frac{q^2-1}{6} + (t+1)(s+1)\right] - (p-1)\left[\frac{q^2-1}{3} + (t+1)(s+1)\right]$$

$$= 2ps + (p-1)(t+2)s + 2s(2q-1) > 0.$$

Lemma 2.5 Let G_0 be a bicyclic graph with $V(G) = \{v_1, \dots, v_s\} \cup V(C_p) \cup V(C_q)$, for which u is a vertex of degree s in the cycle C_q of the bicyclic graph G_0 , and uv_1, uv_2, \dots, uv_s are pendent edges incident with u . The other cycle C_p only has one common vertex w with C_q . Let graph G_1 delete the edges uv_1, uv_2, \dots, uv_s , and add new edges wv_1, wv_2, \dots, wv_s . For convenience, let $r(w, u) = t$. Then $D_R(G_0) > D_R(G_1)$.

Proof. Let H_0 be a subgraph $G[\{v_1, \dots, v_s\} \cup V(C_q)]$ and $S = G[\{u, v_1, \dots, v_s\}]$ (H_1 be a subgraph $G_1[\{v_1, \dots, v_s\} \cup V(C_q)]$ and $S_1 = G'[\{w, v, v_1, \dots, v_s\}]$, respectively). By Lemmas 2.2 and 2.3, we get

$$\begin{aligned}
D_R(G_0) &= D_R(C_p) + D_R(H_0) + 2(s+q)r(w|C_p) + 2pr(w|H_0) + (q+s-1)S'(w|C_p) + (p-1)S'(w|H_0), \\
D_R(H_0) &= D_R(C_q) + D_R(S) + 2sr(u|C_q) + 2qr(u|S) + sS'(u|C_q) + (q-1)S'(u|S), \\
D_R(G_1) &= D_R(C_p) + D_R(H_1) + 2(s+q)r(w|C_p) + 2pr(w|H_1) + (q+s-1)S'(w|C_p) + (p-1)S'(w|H_1), \\
D_R(H_1) &= D_R(C_q) + D_R(S_1) + 2sr(w|C_q) + 2qr(w|S_1) + sS'(w|C_q) + (q-1)S'(w|S_1), \\
D_R(G_0) - D_R(G_1) &= 2pr(w|H_0) + (p-1)S'(w|H_0) - 2pr(w|H_1) - (p-1)S'(w|H_1) \\
&= 2p\left[\frac{q^2-1}{6} + (t+1)s\right] + (p-1)\left[\frac{q^2-1}{3} + (t+1)s\right] \\
&\quad - 2p\left[\frac{q^2-1}{6} + s\right] - (p-1)\left[\frac{q^2-1}{3} + s\right] \\
&= (3p-1)ts > 0.
\end{aligned}$$

Definition 2.2 Let G be a bicyclic graph G with $V(G) = \{u, v, v_1, \dots, v_s\} \cup V(C_p) \cup V(C_q)$, for which v is a vertex of degree $s+1$ such that vv_1, vv_2, \dots, vv_s are pendent edges incident with v , and u is the neighbor of v distinct from v_i that is on the cycle C_q . The other cycle C_p only has one common vertex w with C_q . We form a graph $G'' = \pi(G, v)$ by deleting the edges vv_1, vv_2, \dots, vv_s and connecting v_i and all the isolated vertices into a path $vv_1 \dots v_s$. We say that G'' is a π -transform of the graph G , see Fig. 4.

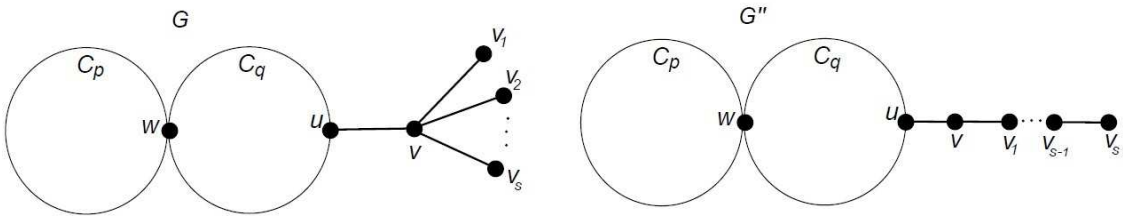


Fig. 4. π -transform of a vertex v .

Lemma 2.6 Consider the graphs defined in Definition 2.1 and let $G'' = \pi(G, v)$ be a π -transform of the bicyclic graph G , for convenience, let $r(w, u) = t$. Then $D_R(G) < D_R(G'')$.

Proof. Let H be a subgraph $G[\{u, v, v_1, \dots, v_s\} \cup V(C_q)]$ and $T = G[\{u, v, v_1, \dots, v_s\}]$ (H'' be a subgraph $G''[\{u, v, v_1, \dots, v_s\} \cup V(C_q)]$ and $P = G'[\{u, v, v_1, \dots, v_s\}]$, respectively). By Lemmas 2.2 and 2.3, we get

$$\begin{aligned}
D_R(G) &= D_R(C_p) + D_R(H) + 2(s+1+q)r(w|C_p) + 2pr(w|H) + (q+s)S'(w|C_p) + (p-1)S'(w|H), \\
D_R(H) &= D_R(C_q) + D_R(T) + 2(s+1)r(u|C_q) + 2qr(u|T) + (s+1)S'(u|C_q) + (q-1)S'(u|T), \\
D_R(G'') &= D_R(C_p) + D_R(H'') + 2(s+1+q)r(w|C_p) + 2pr(w|H'') + (q+s)S'(w|C_p) + (p-1)S'(w|H''), \\
D_R(H'') &= D_R(C_q) + D_R(P) + 2(s+1)r(u|C_q) + 2qr(u|P) + (s+1)S'(u|C_q) + (q-1)S'(u|P), \\
D_R(G) - D_R(G'') &= D_R(T) + 2qr(u|T) + (q-1)S'(u|T) + 2pr(w|H) + (p-1)S'(w|H) \\
&\quad - D_R(P) - 2qr(u|P) - (q-1)S'(u|P) - 2pr(w|H'') - (p-1)S'(w|H'') \\
&= (3s^2 - s) + 2q(2s+1) + (q-1)(s+1+2s) + 2p\left[\frac{q^2-1}{6} + t+1 + (t+2)s\right] \\
&\quad + (p-1)\left[\frac{q^2-1}{3} + (t+1)(s+1) + (t+2)s\right] - \left(\frac{2}{3}s^3 + s^2 + \frac{1}{3}\right) \\
&\quad - 2q\frac{(1+s+1)(s+1)}{2} - (q-1)\left[(s+1)s + s+1\right] \\
&\quad - 2p\left[\frac{q^2-1}{6} + t(s+1) + \frac{(1+s+1)(s+1)}{2}\right] \\
&\quad - (p-1)\left[\frac{q^2-1}{3} + (t+1+t+s)s + t+s+1\right] \\
&= (s-s^2)(2q+2p-2) - \frac{2}{3}s(s-1)(s-2) < 0.
\end{aligned}$$

Lemma 2.7 Let G_0 be a bicyclic graph with the vertex set $V(C_p) \cup V(C_q) \cup V(P_{s+1})$, in which $V(C_p) \cap V(P_{s+1}) = \{v\}$ and $V(C_q) \cap V(P_{s+1}) = \{w\}$. For $wa \in E(P_{s+1})$ and $u \in V(C_q)$, let $G_1 = (G - \{aw\}) \cup \{ua\}$. For convenience, let $r(u, w) = t$. Then $D_R(G_0) > D_R(G_1)$.

Proof. By Lemmas 2.2 and 2.3, we get

$$\begin{aligned}
D_R(G_0) &= D_R(H_0) + D_R(H) + 2(p+s-1)r(a|H_0) + 2(q+1)r(a|H) + (p+s-2)S'(a|H_0) + qS'(a|H), \\
D_R(G_1) &= D_R(H_1) + D_R(H) + 2(p+s-1)r(w|H_1) + 2(q+1)r(w|H) + (p+s-2)S'(w|H_1) + qS'(w|H), \\
D_R(G_0) - D_R(G_1) &= 2(p+s-1)r(a|H_0) + (p+s-2)S'(a|H_0) \\
&\quad - 2(p+s-1)r(w|H_1) - (p+s-2)S'(w|H_1) \\
&= 2(p+s-1)\left(1 + \frac{q^2-1}{6} + q\right) + (p+s-2)\left(\frac{q^2-1}{3} + 3 + 2q\right) \\
&\quad - 2(p+s-1)\left(\frac{q^2-1}{6} + t+1\right) - (p+s-2)\left(\frac{q^2-1}{3} + 2t+1\right) \\
&= 2(p+s-1)(q-t) + (p+s-2)(2q-2t+2) > 0.
\end{aligned}$$

3 Main results

In this section, we will characterize n -vertex bicyclic graphs with exactly two cycles having minimum and maximum degree resistance distances.

Theorem 3.1 *Let $G \in \mathcal{B}_n^{p,q}$ and $G \neq S_n^{p,q}$. Then $D_R(G) > D_R(S_n^{p,q})$.*

Proof. Suppose that a bicyclic graph G_0 has minimal degree resistance distance among graphs in $\mathcal{B}_n^{p,q}$. For G_0 , we prove the following claims.

Claim 1. In Fig. 1, T_{v_i}, T_{u_j} and T_{w_k} are all stars with their centers at v_i, u_j and w_k for each i, j and k .

Without loss of generality, suppose that tree T_{v_i} is not a star. Let G_1 be constructed from G_0 by deleting all the edges of T_{v_i} and connecting all the isolated vertices to v_i ; that is, T_{v_i} is a star in G_1 with its center at v_i and denote it by S_{v_i} . By Lemma 2.4, $D_R(G_0) > D_R(G_1)$, which contradicts the choice of G_0 . Hence Claim 1 holds. ■

Claim 2. The length of P is 0.

Suppose to the contrary that the length of P is $k(k \geq 1)$. Assume that $v_1 = w_0, u_1 = w_k$. Let $e = w_i w_{i+1}$ be an edge of P . Let G_2 be the graph obtained from G_0 by first contracting e and then attaching a pendent edge $w_i a$ to w_i . Assume that G_{01} and G_{02} are two components of $G_0 - e$ and G_{21} and G_{22} are copies of G_{01} and G_{02} in G_2 , respectively. See Fig. 5.

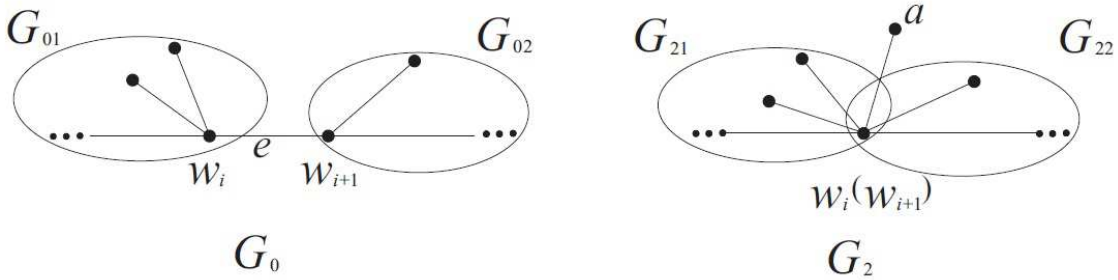


Fig. 5. Graphs G_0 and G_2 .

In the following, we prove $D_R(G_2) < D_R(G_0)$.

Proof. By Lemma 2.3, we get

$$\begin{aligned} D_R(G_0) &= D_R(G_{01}) + D_R(G_{02} + w_i w_{i+1}) + 2(m_{02} + 1)r(w_i | G_{01}) + 2m_{01}r(w_i | G_{02} + w_i w_{i+1}) \\ &\quad + n_{02}S'(w_i | G_{01}) + (n_{01} - 1)S'(w_i | G_{02} + w_i w_{i+1}), \end{aligned}$$

$$\begin{aligned} D_R(G_2) &= D_R(G_{21}) + D_R(G_{22} + w_i a) + 2(m_{22} + 1)r(w_i | G_{21}) + 2m_{21}r(w_i | G_{22} + w_i a) \\ &\quad + n_{22}S'(w_i | G_{21}) + (n_{21} - 1)S'(w_i | G_{22} + w_i a), \end{aligned}$$

$$\begin{aligned}
D_R(G_0) - D_R(G_2) &= 2m_{01}r(w_i|G_{02} + w_iw_{i+1}) + (n_{01} - 1)S'(w_i|G_{02} + w_iw_{i+1}) \\
&\quad - 2m_{21}r(w_i|G_{22} + w_ia) - (n_{21} - 1)S'(w_i|G_{22} + w_ia) \\
&= 2m_{01}\left[r(w_{i+1}|G_{02}) + n + 1\right] + (n_{01} - 1)\left[S'(w_{i+1}|G_{02}) + n + n + 1\right] \\
&\quad - 2m_{21}\left[r(w_{i+1}|G_{22}) + 1\right] - (n_{21} - 1)\left[S'(w_{i+1}|G_{22}) + n + 1\right] \\
&= (2m + n - 1)n > 0.
\end{aligned}$$

We obtain $D_R(G_2) < D_R(G_0)$. This contradicts the hypothesis. Hence Claim 2 holds. \blacksquare

Claim 3. In Fig. 1, if $p + q \leq n$, then only $T_{v_1}(T_{v_1} = T_{u_1})$ is nontrivial.

Without loss of generality, suppose to the contrary that tree $T_{u_i}(i \neq 1)$ is nontrivial. By Lemma 2.5, $D_R(G_0) > D_R(G_1)$, which contradicts the choice of G_0 . Hence Claim 3 holds.

Claims 1-3 yield Theorem 3.1. \blacksquare

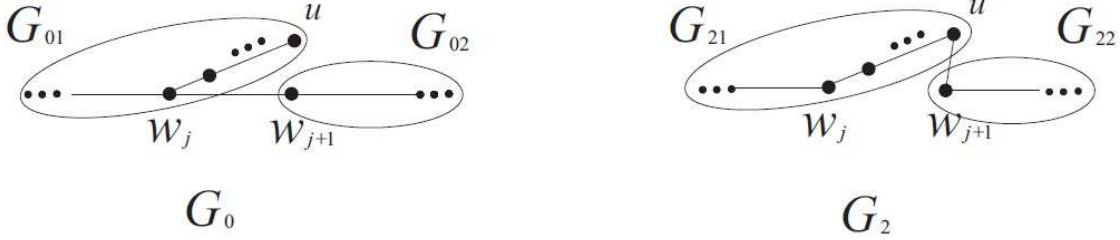


Fig. 6. Graphs G_0 and G_2 .

Theorem 3.2 . Let $G \in \mathcal{B}_n^{p,q}$ and $G \neq P_n^{p,q}$. Then $D_R(G) < D_R(P_n^{p,q})$.

Proof. Suppose that a bicyclic graph G_0 has maximal degree resistance distance among graphs in $\mathcal{B}_n^{p,q}$. For G_0 , we prove the following Claims.

Claim 1. In Fig. 1, T_{v_i}, T_{u_j} and T_{w_k} are all paths with their end vertices v_i, u_j and w_k for each i, j and k .

Without loss of generality, suppose that tree T_{v_i} is not a path. Let G_1 be the graph constructed from G_0 by deleting all the edges of T_{v_i} and connecting v_i and all the isolated vertices into a path; that is, T_{v_i} is a path with end vertex v_i in G_1 and denote it by P_{v_i} . By Lemma 2.6, $D_R(G_1) > D_R(G_0)$, which contradicts the choice of G_0 . Hence Claim 1 holds.

Claim 2. Assume that $T_{w_0} = T_{v_1}$ and $T_{w_m} = T_{u_1}$, then T_{w_i} is trivial ($0 \leq i \leq m$).

If not, without loss of generality, suppose that there is nontrivial T_{w_j} . By Claim 1, we know that T_{w_j} is a path with w_j as its end vertex and assume that u is the other end vertex. Let $G_2 = G_0 - w_jw_{j+1} + uw_{j+1}$ (if $j = m$, $G_2 = G_0 - w_{j-1}w_j + uw_{j-1}$). Assume that G_{01} and G_{02} are two components of $G_0 - w_jw_{j+1}$ and G_{21} and G_{22} are two components of $G_2 - uw_{j+1}$. See Fig. 4.

In the following, we prove $D_R(G_2) > D_R(G_0)$.

Proof. By Lemma 2.3, we get

$$\begin{aligned}
D_R(G_0) &= D_R(G_{01} + w_j w_{j+1}) + D_R(G_{02}) + 2m_{02}r(w_{j+1}|G_{01} + w_j w_{j+1}) + 2(m_{01} + 1)r(w_{j+1}|G_{02}) \\
&\quad + (n_{02} - 1)S'(w_{j+1}|G_{01} + w_j w_{j+1}) + n_{01}S'(w_{j+1}|G_{02}), \\
D_R(G_2) &= D_R(G_{21} + uw_{j+1}) + D_R(G_{22}) + 2m_{22}r(w_{j+1}|G_{21} + uw_{j+1}) + 2(m_{21} + 1)r(w_{j+1}|G_{22}) \\
&\quad + (n_{22} - 1)S'(w_{j+1}|G_{21} + uw_{j+1}) + n_{21}S'(w_{j+1}|G_{22}), \\
D_R(G_{01} + w_j w_{j+1}) &= D_R(G_{01}) + D_R(w_j w_{j+1}) + 2r(w_j|G_{01}) + 2m_{01}r(w_j|w_j w_{j+1}) \\
&\quad + S'(w_j|G_{01}) + (n_{01} - 1)S'(w_j|w_j w_{j+1}), \\
D_R(G_{21} + uw_{j+1}) &= D_R(G_{21}) + D_R(uw_{j+1}) + 2r(u|G_{21}) + 2m_{21}r(u|uw_{j+1}) \\
&\quad + S'(u|G_{21}) + (n_{21} - 1)S'(u|uw_{j+1}), \\
D_R(G_0) - D_R(G_2) &= 2r(w_j|G_{01}) + S'(w_j|G_{01}) + 2m_{02}r(w_{j+1}|G_{01} + w_j w_{j+1}) \\
&\quad + (n_{02} - 1)S'(w_{j+1}|G_{01} + w_j w_{j+1}) - 2r(u|G_{21}) - S'(u|G_{21}) \\
&\quad - 2m_{22}r(w_{j+1}|G_{21} + uw_{j+1}) - (n_{22} - 1)S'(w_{j+1}|G_{21} + uw_{j+1}) \\
&< 2m_{02} \left[r(w_{j+1}|G_{01}) + 1 \right] + (n_{02} - 1) \left[S'(w_{j+1}|G_{01}) + 3 \right] \\
&\quad - 2m_{22} \left[r(w_{j+1}|G_{21}) + 1 \right] - (n_{22} - 1) \left[S'(w_{j+1}|G_{21}) + 2 \right] \\
&= (n - 1) \left[S'(w_{j+1}|G_{01}) - S'(w_{j+1}|G_{21}) + 1 \right] < 0.
\end{aligned}$$

We obtain $D_R(G_2) > D_R(G_0)$. This contradicts the hypothesis. Hence Claim 2 holds.

Claim 3. In Fig. 1, if $p + q \leq n$, then T_{v_i} and T_{u_j} are trivial for each i and j .

Without loss of generality, suppose to the contrary that $T_{v_i} (i \neq 1)$ is nontrivial. By Lemma 2.7, $D_R(G_0) < D_R(G_1)$, which contradicts the choice of G_0 . Hence Claim 3 holds.

Claims 1-3 yield Theorem 3.2. ■

We now compute $D_R(S_n^{p,q})$ and $D_R(P_n^{p,q})$.

For $S_n^{p,q}$, see Fig. 2(a).

$$\begin{aligned}
D_R(S_n^{p,q}) &= D_R(C_q) + D_R(H) + 2(p + s)r(v_1|C_q) + 2qr(v_1|H) + (p + s - 1)S'(v_1|C_q) + (q - 1)S'(v_1|H), \\
D_R(H) &= D_R(C_p) + D_R(S) + 2sr(v_1|C_p) + 2pr(v_1|S) + sS'(v_1|C_p) + (p - 1)S'(v_1|S), \\
r(v_1|H) &= r(v_1|C_p) + r(v_1|S), \\
S'(v_1|H) &= S'(v_1|C_p) + S'(v_1|S), \\
D_R(S_n^{p,q}) &= \frac{q^3 - q}{3} + \frac{p^3 - p}{3} + 3(s + 1)^2 - 7(s + 1) + 4 + 2s\frac{p^2 - 1}{6} + 2ps \\
&\quad + s\frac{p^2 - 1}{3} + (p - 1)s + 2(p + s)\frac{q^2 - 1}{6} + 2q\frac{p^2 - 1}{6} + 2qs \\
&\quad + (p + s - 1)\frac{q^2 - 1}{3} + (q - 1)\frac{p^2 - 1}{3} + (q - 1)s \\
&= \frac{1}{3}(q^3 + p^3 + 2qp^2 + 2q^2p + 2sp^2 + 2sq^2 + 9sp + 9sq - q^2 - p^2 - 3q - 3p + 9s^2 - 13s + 2).
\end{aligned}$$

And $s = n + 1 - p - q$, so we get

$$D_R(S_n^{p,q}) = \frac{1}{3}[-p^3 - q^3 + (2n + 1)(p^2 + q^2) + (1 - 9n)(p + q) + 9n^2 + 5n - 2]. \quad (1)$$

For $P_n^{p,q}$, see Fig. 2(b).

$$\begin{aligned} D_R(P_n^{p,q}) &= D_R(C_p) + D_R(H) + 2(q + m)r(v_1|C_p) + 2pr(v_1|H) + (q + m - 1)S'(v_1|C_p) + (p - 1)S'(v_1|H), \\ D_R(H) &= D_R(C_q) + D_R(P_{m+1}) + 2qr(u_1|P_{m+1}) + 2mr(u_1|C_q) + (q - 1)S'(u_1|P_{m+1}) + sS'(u_1|C_q), \end{aligned}$$

$$r(v_1|H) = r(v_1|C_q) + r(v_1|P_{m+1}),$$

$$S'(v_1|H) = S'(v_1|C_q) + S'(v_1|P_{m+1}),$$

$$\begin{aligned} D_R(P_n^{p,q}) &= \frac{q^3 - q}{3} + \frac{p^3 - p}{3} + \frac{2}{3}m^3 + m^2 + \frac{1}{3}m + 2q\frac{(1 + m)m}{2} + 2m\frac{q^2 - 1}{6} + (q - 1)m^2 + \frac{q^2 - 1}{3}m \\ &\quad + 2(q + m)\frac{p^2 - 1}{6} + 2p[\frac{(1 + m)m}{2} + m(q - 1) + \frac{q^2 - 1}{6}] \\ &\quad + (q + m - 1)\frac{p^2 - 1}{3} + (p - 1)[(1 + m - 1)(m - 1) + 3m + 2m(q - 1) + \frac{q^2 - 1}{3}] \\ &= \frac{1}{3}\left[p^3 + q^3 + (2q + 2m - 1)p^2 + (2p + 2m - 1)q^2 + (6m^2 - 3m - 3)(p + q) \right. \\ &\quad \left. + 12mpq + 2m^3 - 3m^2 - 3m + 2\right]. \end{aligned}$$

And $m = n + 1 - p - q$, so we get

$$D_R(P_n^{p,q}) = \frac{1}{3}\left[3p^3 + 3q^3 + (4n + 5)(p^2 + q^2) + (3n + 3)(p + q) + 2n^3 + 3n^2 - 3n - 2\right]. \quad (2)$$

■

4 Bicyclic graphs with extremal degree resistance distance

By Theorems 3.1 and 3.2, n -vertex bicyclic graphs in $\mathcal{B}_n^{p,q}$ with minimal and maximal degree resistance distance must belong to the classes of $S_n^{p,q}$ and $P_n^{p,q}$, respectively. In what follows, we will determine those which has the extremal degree resistance distance among graphs in $\mathcal{B}_n^{p,q}$.

Theorem 4.1 *Among all n -vertex bicyclic graphs, the graph $S_n^{3,3}$ has the minimum degree resistance distance.*

$$D_R(S_n^{3,3}) = 3n^2 - \frac{13}{3}n - \frac{32}{3}, \quad n \geq 5.$$

Proof. Let u_1, u_2, w be three successive vertices lying on the cycle C_p of the bicyclic graph G_1 . The other cycle C_q only has one common vertex w with C_p . And wv_1, wv_2, \dots, wv_s are pendent edges incident with w . Let the graph G_2 is obtained by deleting the edges u_1u_2 and adding the edge wu_2 . Then $D_R(G_2) < D_R(G_1)$.

$$D_R(G_1) = D_R(C_p) + D_R(H_1) + 2pr(w|H_1) + 2(s + q)r(w|C_p) + (p - 1)S'(w|H_1) + (q + s - 1)S'(w|C_p),$$

$$\begin{aligned}
D_R(G_2) &= D_R(C'_p) + D_R(H_1) + 2pr(w|H_1) + 2(s+q)r(w|C'_p) + (p-1)S'(w|H_1) + (q+s-1)S'(w|C'_p), \\
D_R(G_2) - D_R(G_1) &= D_R(C'_p) + 2(s+q)r(w|C'_p) + (q+s-1)S'(w|C'_p) \\
&\quad - D_R(C_p) - 2(s+q)r(w|C_p) - (q+s-1)S'(w|C_p) \\
&= D_R(C_{p-1}) + D_R(u_1w) + 2(p-1)r(w|u_1w) + 2r(w|C_{p-1}) \\
&\quad + (p-1)S'(w|u_1w) + S'(w|C_{p-1}) - D_R(C_p) + 2(s+q)r(w|C'_p) \\
&\quad - 2(s+q)r(w|C_p) + (s+q-1)S'(w|C'_p) - (s+q-1)S'(w|C_p) \\
&= \frac{(p-1)^3 - (p-1)}{3} + 2 + 2p - 2 + 2\frac{(p-1)^2 - 1}{6} + p - 1 \\
&\quad + \frac{(p-1)^2 - 1}{3} - \frac{p^3 - p}{3} + 2(s+q)\left[\frac{(p-1)^2 - 1}{6} + 1 - \frac{p^2 - 1}{6}\right] \\
&\quad + (s+q-1)\left[\frac{(p-1)^2 - 1}{3} + 1 - \frac{p^2 - 1}{3}\right] \\
&= -\frac{1}{3}p^2 - \frac{8}{3}p - 1 - \frac{4}{3} + \frac{2}{3}p + \frac{s}{3}(11 - 4p) + \frac{q}{3}(11 - 4p) \\
&= -\frac{1}{3}p^2 - 2p - \frac{7}{3} + \frac{s}{3}(11 - 4p) + \frac{q}{3}(11 - 4p) < 0.
\end{aligned}$$

Combining the above discussions, Eq. (1) and $p = q = 3$, we can get

$$D_R(S_n^{3,3}) = 3n^2 - \frac{13}{3}n - \frac{32}{3}, \quad n \geq 5.$$

■

Theorem 4.2 *Among all n -vertex bicyclic graphs, the graph $P_n^{3,3}$ has the maximal degree resistance distance.*

$$D_R(P_n^{3,3}) = \frac{2}{3}n^3 + n^2 - 19n + \frac{88}{3}, \quad n \geq 5.$$

Proof. Let u_1, w, u_2 be three successive vertices lying on the cycle C_p of the bicyclic graph G_3 . The cycles C_p and C_q are linked with two end vertexes v and w of P_{s+1} . Let the graph G_4 is obtained by deleting the edge wu_2 and adding the edge u_1u_2 . Then $D_R(G_4) > D_R(G_3)$.

$$\begin{aligned}
D_R(G_3) &= D_R(C_p) + D_R(H) + 2(q+s)r(w|C_p) + 2pr(w|H) + (q+s-1)S'(w|C_p) + (p-1)S'(w|H), \\
D_R(G_4) &= D_R(C'_p) + D_R(H) + 2(q+s)r(w|C'_p) + 2pr(w|H) + (q+s-1)S'(w|C'_p) + (p-1)S'(w|H), \\
D_R(C'_p) &= D_R(C_{p-1}) + D_R(wu_1) + 2r(u_1|C'_p) + 2(p-1)r(u_1|wu_1) + S'(u_1|C'_p) + (p-2)S'(u_1|wu_1) \\
&= \frac{(p-1)^3 - (p-1)}{3} + 2 + 2\frac{(p-1)^2 - 1}{6} + 2(p-1) + \frac{(p-1)^2 - 1}{3} + (p-2) \\
&= \frac{p^3}{3} + \frac{p^2}{3} + \frac{8p}{3} + \frac{7}{3}, \\
r(w|C'_p) &= 1 + \frac{(p-1)^2 - 1}{3} + (p-1) = \frac{p^2}{6} + \frac{2p}{3}, \\
S'(w|C'_p) &= 3 + \frac{(p-1)^2 - 1}{3} + 2(p-1) = \frac{p^2 - 2p}{3} + 2p + 1,
\end{aligned}$$

$$\begin{aligned}
D_R(G_4) - D_R(G_3) &= D_R(C'_p) + 2(q+s)r(w|C'_p) + (q+s-1)S'(w|C'_p) \\
&\quad - D_R(C_p) - 2(q+s)r(w|C_p) - (q+s-1)S'(w|C_p) \\
&= \frac{p^3}{3} + \frac{p^2}{3} + \frac{8p}{3} - \frac{7}{3} + 2(q+s)(\frac{p^2}{6} + \frac{2p}{3}) \\
&\quad + (q+s-1)(\frac{p^2-2p}{3} + 2p+1) - \frac{p^3-p}{3} \\
&\quad - 2(q+s)\frac{p^2-1}{6} - (q+s-1)\frac{p^2-1}{3} \\
&= \frac{p^3}{3} + 3p + 2(q+s)(\frac{2p}{3} + \frac{1}{6}) + (q+s-1)(\frac{4p}{3} + \frac{4}{3}) - \frac{7}{3} > 0.
\end{aligned}$$

Combining above discussion, and by Eq. (2) and $p = q = 3$. So we can get

$$D_R(P_n^{3,3}) = \frac{2}{3}n^3 + n^2 - 19n + \frac{88}{3}, \quad n \geq 5.$$

■

5 Concluding remarks

In this paper, we completely characterized the bicyclic graphs with exactly two cycles having extremal degree resistance distances. Although our research is restricted to a small family of graphs, it motivates one to further extend this to the larger classes of graphs. For example, can one determine the maximum or minimum degree resistance distance for tricyclic graphs or any general graphs? We plan to investigate this question for larger classes of graphs for future research in this area.

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